Uniformly frustrated XY model without vortex-pattern ordering

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The uniformly frustrated XY model with $f = 1/3$ on a dice lattice is shown to possess a so well developed accidental degeneracy of its ground states that the difference between the free energies of fluctuations does not lead to the stabilization of a particular vortex pattern down to zero temperature. Nonetheless, at low temperatures the system is characterized by a finite helicity modulus whose vanishing (at a finite temperature) is related with the dissociation of half-vortex pairs.

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It is well known that in the presence of external magnetic field a regular two-dimensional Josephson junction array can be described by the Hamiltonian of a uniformly frustrated XY model [1]:

$$H = -J\sum_{(\mathbf{jk})}\cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{j}} - A_{\mathbf{jk}}) \;, \tag{1}$$

where J is the Josephson coupling constant, the fluctuating variables $\varphi_{\mathbf{j}}$ are the phases of the order parameter on superconducting grains forming the array, the quenched variables $A_{\mathbf{jk}}$ are determined by the vector potential of the field and the summation is taken over all pairs of grains connected by a junction. The form of Eq. (1) assumes that the currents in the array are sufficiently small, so their proper magnetic fields can be neglected.

The directed sum of the variables $A_{\mathbf{jk}} \equiv -A_{\mathbf{kj}}$ along the perimeter of each plaquette should be equal to f, the ratio of the magnetic flux per plaquette to the flux quantum Φ_0. It is sufficient to consider the interval $f \in [0, \frac{1}{2}]$, because all other values of f can be reduced to this interval by a simple replacement of variables [1]. The case of $f = 0$ corresponds to the absence of frustration.

For rational f the nature of the vortex pattern in the low-temperature phase of a uniformly frustrated XY model is usually unambiguously determined by the structure of its ground states. The best known examples belonging to this class are the models with $f = 1/2$ and square or triangular lattice [2]. On the other hand, quite often the ground states of a uniformly frustrated XY model are characterized by an accidental degeneracy not related to symmetry [3, 4]. In such cases the nature of vortex ordering at low temperatures cannot be determined without comparing the free energies of fluctuations in the vicinities of different degenerate ground states [5].

Quite often (for example, when the lattice is triangular and $f = 1/3$ or $1/4$ [4]), this mechanism of the removal of an accidental degeneracy works already at the harmonic level, but in some situations (at $f = 1/2$ on honeycomb, dice and *kagome* lattices) one has to take into account the anharmonicites [6]. However, in all the cases investigated insofar, at low enough temperatures a particular periodic vortex pattern can be expected to be stabilized by fluctuations (in the thermodynamic limit).

In the present work we demonstrate that the uniformly frustrated XY model with $f = 1/3$ and dice lattice (see Fig. 1) has rather unique properties. Namely, this is the first example of a frustrated XY model in which the accidental degeneracy of ground states is so well developed that vortex pattern remains disordered at arbitrarily low temperature, although the free energies of fluctuations are different for different periodic patterns. As a consequence, the only phase transition which takes place in this system with the increase of temperature is related to dissociation of pairs of fractional vortices with topological charges $\pm 1/2$. A possibility for the stabilization of a specific periodic pattern can appear only if one goes beyond the limits of the XY model. In the conclusion we briefly discuss the removal of the accidental degeneracy by magnetic interactions and its possible consequences.

The interest to magnetically frustrated superconducting systems with a dice lattice have been motivated by the unusual properties of a single electron spectrum in this geometry [7]. In recent years superconducting wire networks and Josephson junction arrays with a dice lattice have become the subject of active experimental investigations [8, 9, 10] and numerical simulations [11]. Magnetically frustrated Josephson junction arrays formed by rhombic plaquettes have been also discussed in the context of creation of topologically protected qubits [12].

Since both $\varphi_{\mathbf{j}}$ and $A_{\mathbf{jk}}$ depend on a particular choice of the gauge, it is more convenient to describe different states of the system in terms of the gauge-invariant phase differences

$$\theta_{\mathbf{jk}} = \varphi_{\mathbf{k}} - \varphi_{\mathbf{j}} - A_{\mathbf{jk}} \equiv -\theta_{\mathbf{kj}} \;,$$

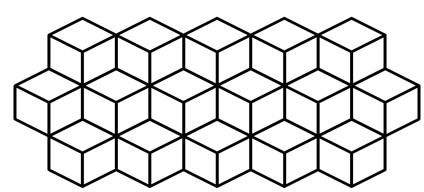

FIG. 1: Dice lattice is periodic and consists of identical rhombic plaquettes with three different orientations.

which below are always assumed to be reduced to the interval $(-\pi, \pi)$. The subscripts $\mathbf{j}$ and $\mathbf{k}$ will be used to denote six- and threefold coordinated sites respectively. If one defines the variable $m_{\mathbf{jj}'}$ to be given by the directed sum of variables θ over the perimeter of the plaquette $\langle \mathbf{jkj}'\mathbf{k}'\rangle$ divided by 2π, the plaquettes for which $m_{\mathbf{jj}'}$ is equal to $1-f$ (rather then to $-f$) are usually referred to as containing vortices. Different local minima of (1) can be then classified by specifying the positions of vortices, whose concentration for $f = 1/3$ should be exactly equal to one third.

The remarkable property of the considered XY model is that all minima of (1) corresponding to different vortex configurations in which vortices do not occupy adjacent plaquettes have the same energy (per bond), $E_0 = -(2/3)J$. Some of these states are schematically shown in Fig. 2. In all such states the variables θ are equal to $\pi/3$ on all bonds surrounding a vortex and to zero on all other bonds (shared by two plaquettes without vortices). It is evident that these states correspond to the absolute minimum of energy, because they minimize the energy separately for each plaquette containing a vortex and for each of the remaining bonds. The barriers between the "neighboring" ground states have the height $E_\mathrm{b} = 6(2-\sqrt{3})J \approx 1.61J$.

All these ground states [whose degeneracy survives even if the interaction in (1) deviates from cosine] can be put into correspondence with the ground states of the antiferromagnetic Ising model defined on the triangular lattice $\mathcal{T}$ formed by the sixfold coordinated sites $\{\mathbf{j}\}$. In all ground states of such a model each triangular plaquette should contain exactly one bond with parallel spins ($s_\mathbf{j}s_{\mathbf{j}'} = 1$) [13]. The existence of a mapping becomes evident as soon as one notices that in all ground states of the considered XY model each hexagon formed by three neighboring plaquettes should contain exactly one vortex. The plaquettes with vortices can be then identified with the bonds connecting parallel spins by setting $s_\mathbf{j}s_{\mathbf{j}'} = 2m_{\mathbf{jj}'} - 1/3$.

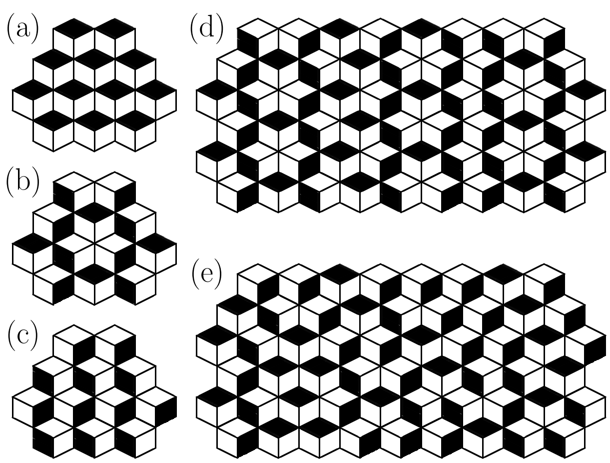

FIG. 2: Some ground states of the frustrated XY model with a dice lattice and $f = 1/3$. The black plaquettes correspond to $m = 2/3$ and white plaquettes to $m = -1/3$.

At zero temperature, $T = 0$, the antiferromagnetic Ising model on a triangular lattice is characterized by an algebraic decay of correlation functions [14] and a finite extensive entropy, which is related to the possibility of creation of zero-energy domain walls forming closed loops [13]. The same partition function can be interpreted as the partition function of the SOS (solid-on-solid) model suitable for the description of the height fluctuations on the (111) facet of a crystal with a simple cubic lattice [15]. In terms of this SOS model zero-energy domain walls correspond to zero-energy steps, and the algebraic correlations of spins are translated into logarithmic correlations of integer variables $h_\mathbf{j}$ and $n_\mathbf{k}$, which can be associated with the height of the surface. These variables can be introduced following the relations

$$h_{\mathbf{j}\pm\mathbf{e}_\alpha} = h_\mathbf{j} \pm 3m_{\mathbf{j},\mathbf{j}\pm\mathbf{e}_\alpha}, \quad n_\mathbf{k} = \frac{1}{3}\sum_{\mathbf{j}=\mathbf{j}(\mathbf{k})} h_\mathbf{j} \, ,$$

where $\mathbf{e}_\alpha$ (with $\alpha = 1, 2, 3$) are the three basic vectors ($\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$) of $\mathcal{T}$ and $\mathbf{j}(\mathbf{k})$ are the three nearest neighbors of $\mathbf{k}$ on the dice lattice. According to Ref. 15, for $|\mathbf{k}_1 - \mathbf{k}_2| \gg 1$

$$\langle (n_{\mathbf{k_1}} - n_{\mathbf{k_2}})^2\rangle \propto \frac{9}{\pi^2}\ln|\mathbf{k_1} - \mathbf{k_2}| \, . \tag{2}$$

The form of Eq. (2) demonstrates that the SOS model is in the rough phase and that at $T = 0$ the large-scales fluctuations of n can be described by a continuous Gaussian Hamiltonian,

$$H = \frac{K}{2}\int d^2\mathbf{r}(\nabla n)^2 \, , \tag{3}$$

where the dimensionless effective rigidity K (which is of entropic origin) is equal to $K_0 = \pi/9$ [15]. The phase transition of the SOS model to the smooth phase would take place when $K = \pi/2$ [16], thus the system is situated relatively far from the transition point.

The simplest periodic ground state shown in Fig. 2(a) (the striped state) has been discussed in Ref. 11. In terms of the SOS representation this state has the maximal possible slope, whereas a flat state of the SOS model (with $n_\mathbf{k} = \mathrm{const}$) corresponds to the honeycomb vortex pattern of Fig. 2(b). Fig. 2(d) shows two parallel steps (of opposite signs) which separate flat states with $\Delta n = \pm 1$. If the left step of Fig. 2(d) is repeated as often as possible, one obtains the striped state of Fig. 2(a). On the other hand, the repetition of the right step of Fig. 2(d) leads to the zigzag state shown in Fig. 2(c). However, the steps do not have to be straight, and a typical ground state looks rather disordered, Fig. 2(e).

If the SOS model would be in a smooth phase, the vortex configuration would be characterized by the long-

range order corresponding to the formation of the honeycomb pattern of Fig. 2(b). The fluctuations of $n_{\mathbf{k}}$ and $h_{\mathbf{j}}$ lead to the replacement of the true long-range order by an algebraic decay of correlations of $m_{\mathbf{jj'}}$,

$$\langle m_{\mathbf{j_1 j_1'}} m_{\mathbf{j_2 j_2'}} \rangle \propto |\mathbf{j_1} - \mathbf{j_2}|^{-\eta} \, ,$$

where $\eta = 2$, as follows from the results of Ref. 14. However, one can expect this correlation function to be modulated according to the honeycomb pattern.

At finite temperature, $T > 0$, the equivalence between the considered XY model and the Ising model is no longer exact, because the XY model (i) allows for the existence of continuous fluctuations (spin waves) and (ii) has a more complex classification of topological defects. Numerical calculation of the integrals over the Brillouin zone which determine the free energies of harmonic fluctuations in the vicinity of the three periodic ground states shown in Figs. 2(a), 2(b) and 2(c) reveals that this free energy is the lowest in the honeycomb state [17]. The difference between the free energies of fluctuations in the zigzag and honeycomb states (normalized per single site of $\mathcal{T}$) is given by γT, where $\gamma \approx 2.27 \cdot 10^{-3}$.

Since in terms of the SOS model the zigzag state can be considered as the sequence of steps with the unit density, the same quantity can be also used as an estimate for the effective energy of a step per elementary segment, $E_{\rm st} \approx \gamma T$. The positiveness of $E_{\rm st}$ should lead to a decrease of fluctuations of $n_{\mathbf{k}}$, but since one always has $E_{\rm st}/T \ll 1$, this decrease has to be relatively small. In terms of (3) the influence of a small step energy, $E_{\rm st} \approx \gamma T$, is translated into a very small ($< 1\%$) correction to K_0, $K = K_0 + (2/\sqrt{3})\gamma$. This definitely leaves the system far from the transition to the smooth phase, and, therefore, cannot lead to any qualitative changes from the zero-temperature behavior.

In terms of the original XY model each step of the SOS model corresponds to a line whose crossing shifts the phase variables $\varphi_{\mathbf{j}}$ by π (with respect to what they would be in the absence of this step) on one of the three sublattice into which the triangular lattice $\mathcal{T}$ can be split. After crossing three steps of the same sign the variables $\varphi_{\mathbf{j}}$ are shifted by π for all three sublattices. In other words, the state which is obtained in such a way differs from the original state by a global rotation of all phases by π. As

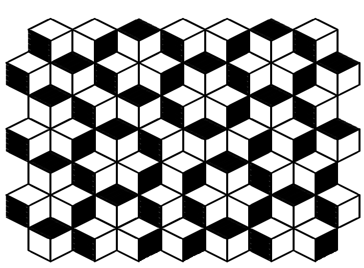

FIG. 3: A possible structure of a half-vortex.

a consequence, the elementary topological excitation of the considered XY model is an object where three steps of the same sign merge together (see Fig. 3) that looks like a screw dislocation with Burgers number $b = \pm 3$, on going around which the phase experiences a continuous rotation by π. For brevity we shall call such defects half-vortices. Since the phase shift by π can be achieved by a phase rotation in both directions, the signs of the two topological charges of a half-vortex (the Burgers number and the vorticity) are not related to each other and can be arbitrary.

The core of a half-vortex can be associated with the three-plaquette cluster which instead of containing exactly one vortex contains either two vortices or no vortices at all (see Fig. 3). In the framework of the antiferromagnetic Ising model the analogous defect cannot exist, because each plaquette can contain only an odd number (1 or 3) of bonds with $s_{\mathbf{j}} s_{\mathbf{j'}} = 1$. The core energy of a half-vortex should be of the order of J.

In accordance with the double nature of half-vortices their interaction consists of two contributions of different origin. Both of them are logarithmic. The first one (the direct interaction) is related to the energy which is required to create the phase twist around the cores and is completely analogous to the interaction of ordinary vortices (with integer vorticity). This interaction is characterized by the prelogarithmic factor $P_{\rm V} = (\pi/2)\Gamma$, where Γ is the helicity modulus of the system [at $T = 0$ in the honeycomb state $\Gamma = \Gamma_0 \equiv (5/4\sqrt{3})J$]. The second contribution is of entropic origin and is related to the interaction of half-vortices as dislocations. It follows from Eq. (3) that for this interaction the prelogarithmic factor is given by

$$P_{\rm D}(b) = \frac{Kb^2}{2\pi} T \, , \tag{4}$$

where one should put $b = 3$.

At low temperatures ($T \ll J$) the direct interaction of half-vortices is dominant, which binds them into small pairs with zero total vorticity. However, the Burgers number of such a pair does not have to be zero, but can be also equal to ± 6. This returns one to the situation in the antiferromagnetic Ising model in which the elementary topological excitations are the dislocations with $b = \pm 6$ [16, 18], the only difference being a slightly larger value of K. Substitution of $K \approx K_0 = \pi/9$ and $b = 6$ in Eq. (4) gives $P_{\rm D}(6) \approx 2$, which is insufficient for such dislocations to be bound in pairs [16]. The application of the Debye-Hückel approximation to the two-dimensional Coulomb gas formed by dislocations shows that $c_{\rm D}$, the concentration of free dislocations, should be exponentially small in $1/T$. The value of $c_{\rm D}$ determines a temperature dependent correlation radius $r_c(T) \propto c_{\rm D}^{-1/2}$, beyond which K is renormalized to zero and even the algebraic correlations of vortex pattern are destroyed.

The presence of free dislocations leads to the screening of the entropic part of the logartihmic interaction of half-vortices at the scales which are large in comparison with $r_c(T)$. Nonetheless, at low temperatures the system will be characterized by a finite value of $\Gamma(T)$, since all half-vortices will be bound in pairs by their direct interaction. With increasing temperature a phase transition will occur related to appearance of free half-vortices and vanishing of $\Gamma(T)$. It will be completely analogous to the Berezinskii-Kosterlitz-Thouless phase transition in the conventional XY model (without frustration), the main difference being that half-vortex pairs dissociate when $T = (\pi/8)\Gamma(T)$ [19], whereas the pairs of ordinary vortices dissociate when $T = (\pi/2)\Gamma(T)$ [20].

Thus, in the present work we have shown that in the frustrated XY model with a dice lattice and $f = 1/3$ the vortex pattern is disordered at any temperature (becoming quasi-ordered only at $T = 0$). Nonetheless, at low temperatures the helicity modulus is finite and jumps to zero only at $T = T_{\rm HV} \sim (\pi/8)\Gamma_0 \approx 0.28\,J$, where the pairs of half-vortices dissociate. This estimate is not far from the value of the transition temperature, $T_c \approx 0.2\,J$, obtained in numerical simulations of Ref. 11.

A possibility of vortex pattern ordering appears only when one goes beyond the limits of the XY model and takes into account some additional mechanism of the removal of an accidental degeneracy. In the case of a proximity coupled array [10] the main role will belong to the energy related to the magnetic fields of currents in the array, which is minimized in the striped state [21]. In terms of the SOS model this leads to a negative step energy, $E_{\rm st} < 0$. In the limit of weak screening, when the corrections to currents from their proper magnetic fields can be neglected, $E_{\rm st} = -\mu J^2/E_\Phi$, where $E_\Phi = \Phi_0^2/4\pi^2 a$ is the characteristic energy, a is the lattice constant (of a dice lattice) and the numerical coefficient μ for both types of steps shown in Fig. 2(d) can be written as

$$\mu \approx \sin^2(\pi/3)\cdot[\lambda_2 - \lambda_4 - 2(\lambda_5 - \lambda_6) + \ldots] \approx 0.25\;,$$

$\lambda_i \equiv -L_i/a > 0$ being the dimensionless values of mutual inductances, L_i, between dice lattice plaquettes [22]. For $a \approx 8\,\mu$m [10] $E_\Phi \approx 10^4\,$K.

With decrease of T the ratio $|E_{\rm st}|/T$ is increased, which for $|E_{\rm st}|/T \ll 1$ will manifest itself only in the decrease of K in Eq. (3). With further decrease of T a phase transition can be expected to occur to a phase with a non-zero slope. In terms of vortices this phase will be characterized by a true long-range order manifesting itself in deviation of occupation probabilities for plaquettes with different orientations from 1/3. This phase transition can be expected to happen when $|E_{\rm st}| \sim T$, *i.e.*, $T/J \sim (\mu T/E_\Phi)^{1/2} \sim 10^{-2}$, and, according to the results of Ref. 23, has to be of the first order. However, at $T/J \lesssim 0.05$ the relaxation of vortex pattern is likely to be dynamically quenched [at $T/J = 0.05$ one has $\exp(-E_{\rm b}/T) \sim 10^{-14}$], which may prevent the observation of such an ordering in experiments or simulations.

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